Sabancı University
Faculty of Engineering and Natural Sciences

EE 314 Digital Communications **Spring 2020**
Assoc. Prof. Mehmet Keskinöz

HOMEWORK #1

Assigned: Feb. 10, 10:00 a.m
Due : Feb. 17, 10:00 a.m.

- ***Remarks: Please keep your answers clear and concise and show all the mathematical derivations that you perform. Each student should write up the solutions entirely on their own. You should list your name and ID on your write-up. If you do not type your solutions in a computer, be sure that your hand-writing is legible, your scan is high-quality and your name and ID are clearly written on your submitted document.***
- ***Your solutions should be scanned as a single pdf file (we wont accept other format such as jpeg or multiple files). You should name your pdf file as first_name_lastname_HW_number (e.g., Mehmet_Keskinoz_HW_1)***
- ***If you have a MATLAB problem, you should also be required to submit your .m file (name .m file as first_name_lastname_HW_number and write your name and ID as a commented header in the .m file) .***
- ***if you don't have a MATLAB related problem in your homework, just upload your solutions as a single pdf file . Otherwise, you should zip your .m file together with your pdf file (name the zip file as as first_name_lastname_HW_number) and upload your single zip file to SUCOURSE.***
- ***Note that you can only get help from your TAs on MATLAB related questions during their office hours.***
- ***If you want to get feedbacks about your homework, you should also submit hand-written (or hard-copy) of your solutions.***
- ***Late submission will not be accepted***

(1) (20 points)

Let us have two continuous-time systems with $x(t)$ at its input and $y(t)$ at its output.

System A: $y(t) = x(t-2) + x(2-t)$

System C: $y(t) = x(t/3)$

Determine which of the following system properties is hold by each of the these continuous-time systems . Justify your answers.

(a) Linear
(b) Time Invariant
(c) Memoryless
(d) Casual
(e) Bounded input-bounded output (BIBO) Stable

Solution:

a) Linearity System A $y(t) = x(t-2) + x(2-t)$

→ $x_1(t) \rightarrow y_1(t) = x_1(t-2) + x_1(2-t)$

→ $x_2(t) \rightarrow y_2(t) = x_2(t-2) + x_2(2-t)$

→ $\alpha x_1(t) + \beta x_2(t) \rightarrow y(t) = [\alpha x_1(t-2) + \beta x_2(t-2)] + [\alpha x_1(2-t) + \beta x_2(2-t)] = \alpha y_1(t) + \beta y_2(t)$

thus, superposition holds, and it is linear ✓ !

system C $y(t) = x(t/3)$

→ $x_1(t) \rightarrow y_1(t) = x_1(t/3)$

→ $x_2(t) \rightarrow y_2(t) = x_2(t/3)$

→ $\alpha x_1(t) + \beta x_2(t) \rightarrow y(t) = \alpha x_1(t/3) + \beta x_2(t/3) = \alpha y_1(t) + \beta y_2(t)$

it is linear ✓ !

b) Time invariance system A

- $x_1(t) \rightarrow y_1(t) = x_1(t-2) + x_1(2-t)$
- $x_2(t) = x_1(t-t_0) \rightarrow y_2(t) = x_1(t-t_0-2) + x_1(2-(t-t_0))$
  $= x_1(t-t_0-2) + x_1(2-t+t_0)$

→ $y_1(t-t_0) = x_1(t-2-t_0) + x_1(2-t-t_0)$

→ $y_2(t) \neq y_1(t-t_0)$ (difference is pink underlined)

⇒ the output is not independent of the time that the input is applied

System is not time invariant ✗ !

system C
- $x_1(t) \rightarrow y_1(t) = x_1(t/3)$
- $x_2(t) = x_1(t-t_0) \rightarrow y_2(t) = x_2(t/3) = x_1\left(\frac{t-t_0}{3}\right)$

→ $y_1(t-t_0) = x_1\left(\frac{t}{3} - t_0\right)$

→ $y_2(t) \neq y_1(t-t_0)$ (difference is circled)

thus, not time invariant ✗ !

c) Memorylessness

system A $y(t) = x(t-2) + x(2-t)$

Output depends on past and future input values, thus not memoryless!

ex: $y(1) = x(-1) + x(1)$ (past value)

ex: $y(-1) = x(-3) + x(3)$ (past value, future value)

system C $y(t) = x(t/3)$

$y(1) = x(1/3)$, $1 > 1/3$ → dependency on past values

$y(-1) = x(-1/3)$, $-1 < -1/3$ → dependency on future values

thus, system is not memoryless!

d) Causality system A

for $t = -1$ → $y(-1) = x(-3) + x(3)$

dependency on future input value, therefore not causal!

system C

for $t = -1$ → $y(-1) = x(-1/3)$, $-1/3 > -1$

dependency on future values, not causal!

e) BIBO Stability "for bounded input values, the output is also bounded"

System A $|x(t)| \leq M_x < \infty, \forall t$ → $|y(t)| \leq M_y = 2M_x < \infty$

BIBO stable!

system C $|x(t)| \leq M_x < \infty, \forall t$ → $|y(t)| \leq M_y = M_x < \infty$

BIBO stable!

* System A: linear, not TI, not memoryless, not causal, BIBO stable//

* System C: linear, not TI, not memoryless, not causal, BIBO stable//

(2) (20 points)

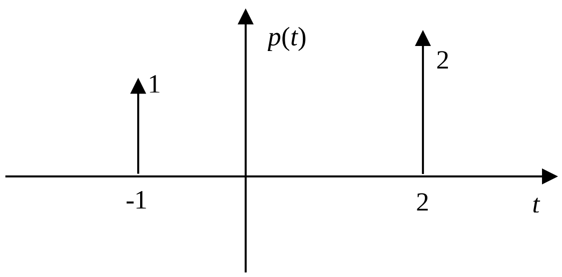
Simplifythe following expressions as much as possible.

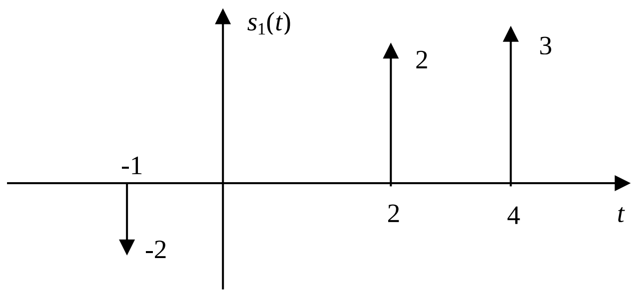
a) $e^{j2\pi t}\left(\delta(t-\frac{1}{4})+\delta(t-\frac{1}{2})+\delta(t-1)+\delta(t+\frac{1}{4})\right)$

b) $\int_{-\infty}^{\infty}\frac{t}{2\tau}\delta(t-2\tau)d\tau$

c) $\int_{-7}^{21}\left(\tan(2t)+e^{-10\pi t}\right)\{\sum_{n=-\infty}^{\infty}\delta(t-8n)\}dt$

d) Determine $y_1(t) = p(t) * s_1(t)$ where * represents the convolution operation, and signals $p(t)$ and $s_1(t)$ are plotted below.





Solution:

a) $e^{j\frac{\pi}{2}}\delta\left(t-\frac{1}{4}\right) + e^{j\pi}\delta\left(t-\frac{1}{2}\right) + e^{j2\pi}\delta\left(t-1\right) + e^{-j\frac{\pi}{2}}\delta\left(t+\frac{1}{4}\right)$

$$= j\,\delta\left(t-\frac{1}{4}\right) - \delta\left(t-\frac{1}{2}\right) + \delta\left(t-1\right) - j\,\delta\left(t+\frac{1}{4}\right)$$

b) $\delta(at) \triangleq \frac{\delta(t)}{|a|}$ $\quad \delta(t-2\tau) = \delta\left(-2\left(\tau - t/2\right)\right) \triangleq \frac{\delta(\tau - t/2)}{2}$

$$\int_{-\infty}^{\infty} \frac{\delta(\tau - t/2)}{2}\cdot\frac{t}{2\tau}\,d\tau = \frac{t}{4}\int_{-\infty}^{\infty}\frac{\delta(\tau - t/2)}{\tau}\,d\tau = \frac{t}{4}\cdot\frac{1}{t/2}$$

$$= 1/2$$

c) $= \int_{-7}^{21} \left(\tan(2t) + e^{-10\pi t}\right)\left(\delta(t) + \delta(t-8) + \delta(t-16)\right)dt$

$$= \int_{-7}^{21} \left(\tan 0 + e^{0}\right)\delta(t) + \left(\tan 16 + e^{-80\pi}\right)\delta(t-8) + \left(\tan 32 + e^{-160\pi}\right)\delta(t-16)\,dt$$

$$= \tan 0 + \tan 16 + \tan 32 + 1 + e^{-80\pi} + e^{-160\pi}$$

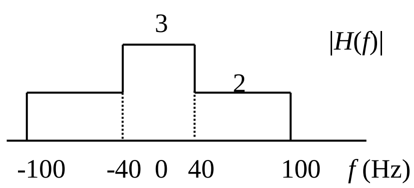
d) $p(t) = \delta(t+1) + 2\,\delta(t-2)$

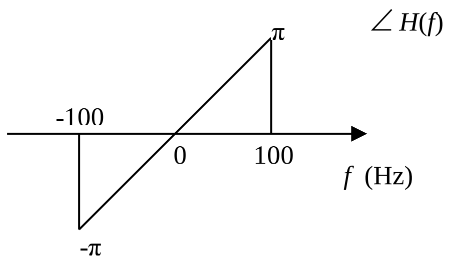
$s_1(t) = -2\,\delta(t+1) + 2\,\delta(t-2) + 3\,\delta(t-4)$

$$p(t) * s_1(t) = -2\,\delta(t+2) - 2\,\delta(t-1) + 3\,\delta(t-3) + 4\,\delta(t-4) + 6\,\delta(t-6)$$

(3)

A filter $H(f)$ has the magnitude (on the left graph) and phase (on the right plot) response shown below. The input $x(t)$ to this filter is $4\cos(30\pi t) - 3\sin(100\pi t) + 7\cos(240\pi t)$. Determine the resulting output.





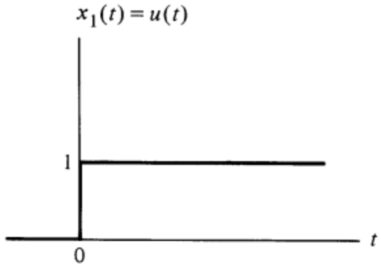
Solution:

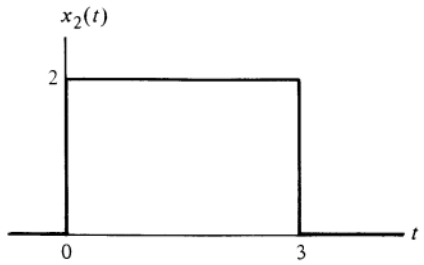
$$x(t) = A\cos(2\pi f_0 t + \phi) \longrightarrow \boxed{\begin{matrix}\text{LTI sys.}\\ H(f)\end{matrix}} \longrightarrow y(t) = A\,|H(f_0)|\cos(2\pi f_0 t + \phi + \measuredangle H(f_0))$$

$$y(t) = 12\cos\left(30\pi t + \frac{15}{100}\pi\right) - 6\sin\left(100\pi t + \frac{\pi}{2}\right)$$

(4)

a) Using convolution, determine and sketch the responses of a linear, time-invariant system with impulse response $h(t) = e^{-t/2}\,u(t)$ to each of the two inputs $x_1(t)$, $x_2(t)$ shown in Figures below. Use $y_1(t)$ to denote the response to $x_1(t)$ and use $y_2(t)$ to denote the response to $x_2(t)$.





b) Express $x_2(t)$ in terms of $x_1(t)$ and then by taking advantage of the linearity and time-invariance properties, determine how $y_2(t)$ can be expressed in terms of $y_1(t)$. Verify your expression by evaluating it with $y_1(t)$ obtained in part (a) and comparing it with $y_2(t)$ obtained in part (a).

Solution

(3) **a.** using the formula for convolution

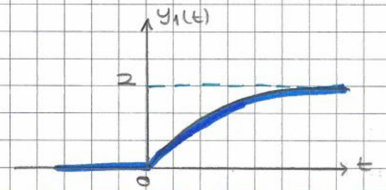
① $y_1(t) = \int_{-\infty}^{+\infty} x(z)\,h(t-z)\,dz = \int_{-\infty}^{+\infty} u(z)\, e^{-\frac{(t-z)}{2}} u(t-z)\,dz$

$= \int_0^t e^{-(t-z)/2}\,dz = +2\,e^{-\frac{(t-z)}{2}}\Big)_0^t = 2\left[e^{\frac{0}{2}} - e^{-\frac{t}{2}}\right]$

$= 2\left(1-e^{-t/2}\right)$ for $t>0$

② $y(t) = 0$ for $\boxed{t<0}$

* we put the upper boundary in the integral as ⓣ, this assumes that $(t>0)$



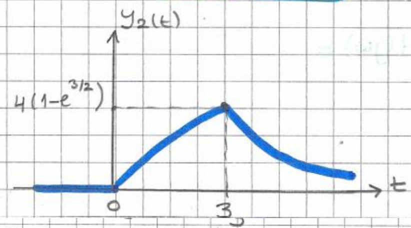
$y_1(t) = \left(2\left[1-e^{-t/2}\right]\right)u(t)$

$y_2(t) = \int^{z} 2\cdot e^{-(t-z)/2}\,dz = \int_0^3 2e^{-(t-z)/2}\,dz = 4e^{-(t-z)/2}\Big)_0^3$

① $= 4e^{-(t-z)/2}\Big)_0^3 = 4\left(e^{-(t-3)/2} - e^{-t/2}\right) = 4e^{-t/2}\left(e^{3/2}-1\right)$ $\boxed{\text{for } t \geq 3}$

② $y_2(t) = \int_0^t 2e^{-(t-z)/2}\,dz = 4\left(1-e^{-t/2}\right)$ $\boxed{\text{for } 0 \leq t \leq 3}$

③ $y_2(t) = 0$ $\boxed{\text{for } t \leq 0}$



b. $x_2(t) = 2\left[x_1(t) - x_1(t-3)\right]$

since the system is LTI:

$x_2(t) \rightarrow \boxed{\text{LTI system}} \rightarrow y_2(t) = 2\left[y_1(t) - y_1(t-3)\right]$

$\Rightarrow$ for $0 \leq t \leq 3$: $y_2(t) = 2\,y_1(t) = 4\left(1-e^{-t/2}\right)$

$\Rightarrow$ for $t \geq 3$: $y_2(t) = 2y_1(t) - 2y_1(t-3) = 4\left(\cancel{1}-e^{-t/2}\right) - 4\left(\cancel{1}-e^{-(t-3)/2}\right)$

$= 4e^{-t/2}\left(e^{3/2}-1\right)$

$\Rightarrow$ for $t<0$: $y_2(t) = 0$

(5) MATLAB exercise about basic signal operations

In this problem we will simulate some signal operations via simulation i.e. scaling, shifting and their combination , where the sampling interval of 0.001 seconds is used. Please write MATLAB codes to perform the following tasks:

i. First generate a segment of an exponentially decaying function

$$y(t) = \exp\left(-\frac{|t|}{4}\right)\left[u(t) - u(t-4)\right]$$

In time interval $[-3 \quad 5]$. Plot $y(t)$

ii. Apply time scaling to generate new signal $y_1(t) = y(2t)$ and plot it

iii. Apply time shifting to generate new signal $y_2(t) = y(t+2)$ and plot it

iv. Apply time scaling, time inversion, and time shifting to generate a new signal $y_3(t) = y(2-2t)$ and plot it

Hints:

- Use $subplot$ command to plot $y(t), y_1(t), y_2(t), y_3(t)$ in a single figure. $\exp()$, $abs()$, $heavyside()$ are some useful functions

Solution:

MATLAB code:

```
syms t
a=heaviside(t)-heaviside(t-4);
y(t)=exp(-abs(t)/4).*a;

subplot(2,2,1);
fplot(y(t),[-5,5]);
title('y(t)');

y1(t)=y(t*2);
subplot(2,2,2);
fplot(y1(t),[-5,5]);
title('y1(t)');

y2(t)=y(t+2);
subplot(2,2,3);
fplot(y2(t),[-5,5]);
title('y2(t)');

y3(t)=y(2-t*2);
subplot(2,2,4);
fplot(y3(t),[-5,5]);
title('y3(t)');
```

Plots for all 4 functions: